A Digital Control Architecture for Quadrotor Aircraft

Nicholas Kottenstette*, Joseph Porter†
*WW Technology Group †ISIS/Vanderbilt University

***Abstract*— We show that the principal attitude and inertial dynamics of a quadrotor aircraft can be decomposed into a cascade of three passive and one interior conic subsystem such that a proportional digital feedback loop can effectively be applied to each subsystem in a nested manner. This proportional feedback architecture includes one saturation block nested between the attitude and inertial control systems to account for actuator saturation. Our architecture can control yaw independently of the desired inertial position. Stability of this architecture can be verified in both simulation and runtime through the following corollary derived from the sector stability theorem of Zames and later Willems. The corollary applies to the control of a dynamic system $H_1 : x_1 \to y_1$ which is inside the sector $[a_1, b_1]$, in which $-\infty < a_1 < 0$, $0 < b_1 \le \infty$, and $b_1 > a_1$. It states that if a negative feedback controller with reference r_1 and control gain $k_1 < -\frac{1}{a_1}$ is applied to $H_1 : x_1 \to y_1$ such that $x_1 = k_1(r_1 - y_1)$ then the closed loop system $H_{cl-1} : r_1 \to y_1$ is L_2^m (l_2^m) stable. Simulations indicate the controller performs exceptionally well when applied to detailed STARMAC and Hummingbird aircraft models which includes blade flapping effects.**

I. INTRODUCTION

Quadrotor aircraft use four rotors to control both lift and body torque for attitude and inertial control. Without a tail rotor and rotors smaller than the primary rotor of a helicopter a quadrotor can achieve higher velocities before blade flapping effects begin to introduce instabilities. However an attitude control system is required for a human to pilot such a vehicle. Thus current research has focused on developing embedded control systems for these quadrotor aircraft.

Building off the earlier work of [1], [2], the authors in [3] present an attitude and height control system with nested saturation blocks for a quadrotor aircraft which achieves asymptotic stability. In [4] a model-independent quaternion-based proportional derivative (PD) attitude controller performs as well as more computationally complex nonlinear controllers. Using backstepping techniques [5] derives image based visual servo control algorithms which exploit passivity-like properties of the dynamic model in order to obtain a Lyapunov stable system. All the above papers, and others contain fairly detailed models which guide their overall control design. Most of the Lyapunov control proposals typically are fairly computationally expensive and it is not clear how robust they are to model uncertainties.

In particular, all of the aforementioned papers appear to neglect a significant time lag characteristic related to the motor thrust command and the corresponding thrust which results due to the acceleration of the air column. With

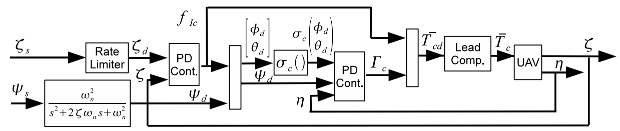

Fig. 1. Proposed quadrotor control system.

[3] as an exception, almost all other model descriptions neglect the limited control thrust due to motor saturation. All model descriptions in previous literature neglect digital platform implementation effects such as sampling delay, quantization, etc. In order to address these effects we propose that Corollary 1 provides a formal test to *verify* that the sector stability condition is satisfied in simulation and during field testing.

As depicted in Fig. 1 we propose to use two PD controllers (denoted as PD Cont. in Fig. 1). The inner-most loop controller is a 'fast' PD attitude controller. The attitude controller's proportional feedback term includes the roll (ϕ), pitch (θ), and yaw ψ Euler angles $\eta = [\phi, \theta, \psi]^\mathsf{T}$. The attitude controller's derivative feedback term y_ω includes the product of the moment of inertia matrix I_b and the body angular velocities ω such that $y_\omega = I_b\omega$. The attitude controller design is initially justified by showing that the dynamic relationships between the body torques Γ and y_ω are passive and assuming that the dynamic relationship between y_ω and η is passive. Next, we further assume that the resulting attitude control system dynamics are 'fast' enough so that we can close the loop with a second PD inertial controller. The proportional feedback term is the inertial position ζ and the derivative feedback term is the inertial velocity $\dot{\zeta} = v_I$. This design is justified because the relationship between the inertial force f_I and velocity v_I is passive as is the relationship from v_I to ζ. Under these preliminary assumptions our control system will result in an overall L_2^m-stable (or bounded) system.

However, there is significant lag between rotor thrust commands and the resulting change in thrust due to the acceleration of the air columns above their respective rotors [6]. A lead compensator denoted as Lead Comp. in Fig. 1) is used to account for the rotor dynamics. The rotors can only apply a fixed range of thrust (denoted $\sigma(\bar{T}_c)$ in which $\bar{T}_c$ denotes the corresponding thrust command vector) due to motor driver voltage limits. It can be inferred from [3] that the relationship between roll (ϕ), pitch (θ), and thrust T to the corresponding desired inertial position can be approximated as a cascade of four integrators subject to input actuator saturation. As a result our linear control law needs

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to be modified with a saturation block in order to achieve asymptotic stability [1, Theorem 2.2]. We do so by limiting the range of the pitch and roll commands to our attitude controller to the interval $[-\frac{\pi}{4}, \frac{\pi}{4}]$ with a saturation function block denoted as $\sigma_c()$ in Fig. 1.

We limit desired inertial position setpoint (denoted as ζ_s) with a position rate change limiter (depicted as 'Rate Limiter' in Fig. 1) in order to avoid destabilizing velocities caused by rotor blade flapping effects. The rate change limiter includes an additional second order filter applied to ζ_s which minimizes overshoot. A similar filter is applied to the yaw set-point ψ_s as well. Other non ideal effects – non passive attitude coordinates (Euler angles η), quantization, floating point math errors, and time delay can be addressed by testing Corollary 1 to *verify* through simulation.

Section II introduces a few definitions regarding passivity, boundedness, and corollaries regarding stability. Section III provides an appropriate model to describe the quadrotor dynamics as it relates to statements regarding the design of a controller for the quadrotor. Section IV provides a description of our control implementation and corresponding stability arguments which lead us to an overall feasible control design. Section V provides a detailed discussion of detailed simulations of our control system used to control a detailed model of the STARMAC and Hummingbird quadrotor aircraft which includes highly nonlinear blade flapping effects [6], [7]. Section VI presents our conclusions and points to future research directions.

II. Passivity and Sector Stability

In order to discuss the (boundedness) or stability properties of the quadrotor with our proposed control system we recall the following nomenclature, definitions and present Corollary 1 in order to *verify* stability. Let $\mathcal{T}$ be the set of times of interest in which $\mathcal{T} = \mathbb{R}^+$ for continuous-time signals and $\mathcal{T} = \mathbb{Z}^+$ for discrete-time signals. Let $\mathcal{V}$ be a linear space $\mathbb{R}^m$ and denote as $\mathcal{H}$ the space of all functions $u : \mathcal{T} \to \mathcal{V}$ which satisfy the following:

$$\|u\|_2^2 = \int_0^\infty u^\mathsf{T}(t)u(t)dt < \infty, \quad (1)$$

for continuous-time systems (L_2^m), and

$$\|u\|_2^2 = \sum_0^\infty u^\mathsf{T}(i)u(i) < \infty, \quad (2)$$

for discrete-time systems (l_2^m). Similarly, we will denote by $\mathcal{H}_e$ the extended space of functions ($u : \mathcal{T} \to \mathcal{V}$) by introducing the truncation operator:

$$x_T(t) = \begin{cases} x(t), & t < T, \\ 0, & t \geq T \end{cases}$$

for continuous time, and

$$x_T(i) = \begin{cases} x(i), & i < T, \\ 0, & i \geq T \end{cases}$$

for discrete time. The extended space $\mathcal{H}_e$ satisfies the following:

$$\|u_T\|_2^2 = \int_0^T u^\mathsf{T}(t)u(t)dt < \infty;\ \forall T \in \mathcal{T} \quad (3)$$

for continuous time systems (L_{2e}^m), and

$$\|u_T\|_2^2 = \sum_0^{T-1} u^\mathsf{T}(i)u(i) < \infty;\ \forall T \in \mathcal{T} \quad (4)$$

for discrete time systems (l_{2e}^m).

Definition 1: A dynamic system $H : \mathcal{H}_e \to \mathcal{H}_e$ is L_2^m stable if

$$u \in L_2^m \implies Hu \in L_2^m. \quad (5)$$

in which $Hu = y$ corresponds to the dynamic output of the system, and the value of Hu at time t will be denoted as $Hu(t) = y(t)$.

Definition 2: A dynamic system $H : \mathcal{H}_e \to \mathcal{H}_e$ is l_2^m stable if

$$u \in l_2^m \implies Hu \in l_2^m. \quad (6)$$

in which $Hu = y$ corresponds to the dynamic output of the system, and the value of Hu at discrete time i will be denoted as $Hu(i) = y(i)$.

The inner product over the interval $[0, T]$ for continuous time is denoted as follows:

$$\langle y, u\rangle_T = \int_0^T y^\mathsf{T}(t)u(t)dt$$

similarly the inner product over the discrete time interval $\{0, 1, \ldots, T-1\}$ is denoted as follows:

$$\langle y, u\rangle_T = \sum_0^{T-1} y^\mathsf{T}(i)u(i).$$

For simplicity of discussion we note the following equivalence for our inner-product space:

$$\langle (Hu)_T, u_T\rangle = \langle (Hu)_T, u\rangle = \langle Hu, u_T\rangle = \langle Hu, u\rangle_T.$$

Definition 3: Assuming that $Hu(0) = y(0) = 0$, then a dynamic system $H : \mathcal{H}_e \to \mathcal{H}_e$ is (strictly) inside the sector $[a, b],\ b > 0,\ a \leq b,\ \epsilon > 0$ if

$$\|y_T\|_2^2 - (a+b)\langle y, u\rangle_T + ab\|u_T\|_2^2 \leq 0\ (\leq -\epsilon\|u_T\|_2^2) \quad (7)$$

Property 1: Assume the following dynamic systems $H : u \to y,\ H_1 : u_1 \to y_1$ are inside their respective sectors $[a, b],\ [a_1, b_1]$, and $k \geq 0$ is a constant then:

(i) I can be said to be inside $[1, 1],\ [\epsilon, 1]\ \forall 0 < \epsilon \leq 1$, or strictly inside $[0, 1+\epsilon]\ \forall 0 < \epsilon \leq 1$.
(ii) kH is inside $[ka, kb]$
(iii) Sum Rule: $(H + H_1)$ is inside $[a + a_1, b + b_1]$.

Definition 4: If we assume that $Hu(0) = 0$, then if H is inside the sector:

i) $[0, \infty]$ it is a passive (positive) system
ii) $[0, b],\ b < \infty$ it is strictly output passive
iii) $[\epsilon, \infty],\ \epsilon > 0$ it is strictly input passive
iv) $[a, b],\ a > 0,\ b < \infty$ it is strictly input-output passive
v) $[a, b],\ -\infty < a,\ b < \infty$ it is a bounded (l_2^m-stable for discrete time, or L_2^m-stable for continuous time) system.

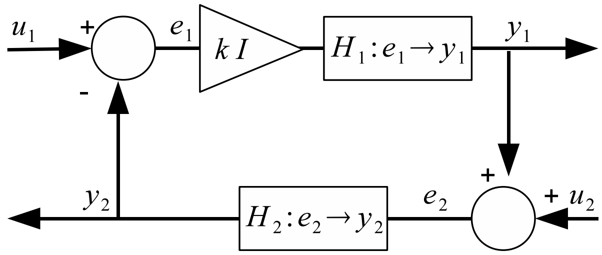

Fig. 2. Bounded system $H : [u_1^\mathsf{T}, u_2^\mathsf{T}]^\mathsf{T} \rightarrow [y_1^\mathsf{T}, y_2^\mathsf{T}]^\mathsf{T}$.

The following Theorem serves as the basis for proposing the linear PD controllers depicted in Fig. 1. It is a weaker form of the passivity theorem [8] which considers when the input $u_2 \neq 0$. Parts of the theorem have appeared in [9]–[12], we generalize it slightly by adding kI to the structure.

Theorem 1: Assume that the combined system $H : u_1 \rightarrow y_1,\ u_2 = 0$ depicted in Fig. 2 has $0 < k < \infty$ and consists of two dynamic systems $H_1 : u_1 \rightarrow y_1$ and $H_2 : u_2 \rightarrow y_2$ which are either:

i) respectively inside the sector $[a_1, b_1],\ a_1 = 0, b_1 < \infty$ (H_1 is strictly output passive) and inside the sector $[0, \infty]$ (H_2 is passive) or
ii) respectively inside the sector $[a_1, b_1],\ a_1 = 0, b_1 = \infty$ (H_1 is passive) and inside the sector $[a_2, \infty]$ in which $a_2 > 0$ (H_2 is strictly input passive),

then $H : u_1 \rightarrow y_1$ is strictly output passive and bounded (l_2^m stable for the discrete time case, or L_2^m stable for the continuous time case).

The following corollary provides a way to verify stability if the closed loop system can not satisfy either of the passivity conditions listed in Theorem 1. In particular we consider the case when H_1 is inside the sector $[a_1, \infty]$ in which $-\infty < a_1 < 0$ [1].

Corollary 1: Assume that the combined dynamic system $H : [u_1^\mathsf{T}, u_2^\mathsf{T}]^\mathsf{T} \rightarrow [y_1^\mathsf{T}, y_2^\mathsf{T}]^\mathsf{T}$ depicted in Fig. 2 consists of two dynamic systems $H_1 : u_1 \rightarrow y_1$ and $H_2 : u_2 \rightarrow y_2$ which are respectively inside the sector $[a_1, b_1]$ and strictly inside the sector $[0, 1+\epsilon],$ for all $\epsilon > 0$. Then H is bounded (L_2^m stable for the continuous time case or l_2^m stable for the discrete time case) if:

$$-\frac{1}{\max\{|a|, b\}} < k < -\frac{1}{a_1}, \quad \text{if } a_1 < 0$$

$$-\frac{1}{b} < k < \infty, \quad \text{otherwise.}$$

III. Quad-rotor Model

Let $\mathcal{I} = \{e_N, e_E, e_D\}$ (North-East-Down) denote the inertial frame, and $\mathcal{A} = \{e_x, e_y, e_z\}$ denote a frame rigidly attached to the aircraft as depicted in Fig. 3. Let ζ denote inertial position, η denote the vector of Euler angles $\eta^\mathsf{T} = \begin{bmatrix}\phi, & \theta, & \psi\end{bmatrix}^\mathsf{T}$ in which ϕ is the roll, θ is the pitch and ψ is the yaw. $R(\eta) = R \in \mathsf{SO}(3)$ is the orthogonal rotation matrix ($R^\mathsf{T} R = I$) which describes the orientation of the airframe in which R describes the rotation matrix from the inertial frame to the body frame as is the convention used in [15], [16]. The rotation matrix allows coordinates relative to the inertial frame such as inertial angular velocity ω_I to coordinates relative to the body frame such as the angular velocity $\omega = [p, q, r]^\mathsf{T}$ as follows

[1] The upper bound on k follows directly from [13, Corollary 4.3.3, case 3] (MIMO) and [14, Theorem 2a, case 2] (SISO). The lower bound on k results from the small gain condition.

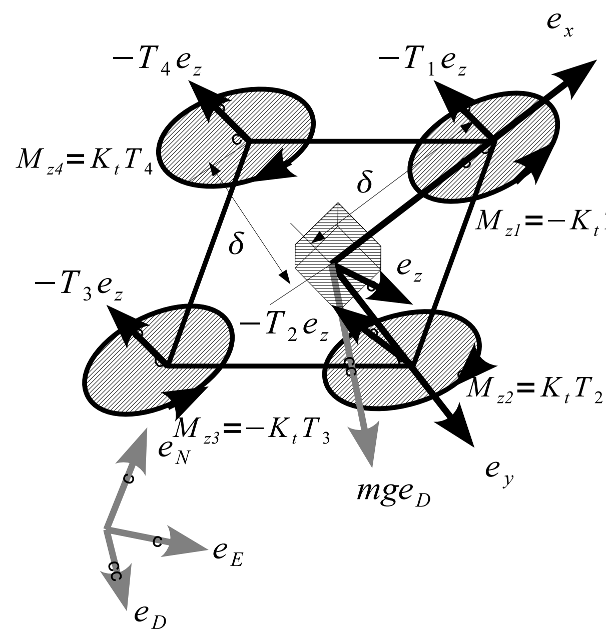

Fig. 3. UAV with depiction of inertial and body frames.

$$\omega_I = R^\mathsf{T} \omega.$$

The standard equations of motion in terms of the aircrafts mass m, and its moment of inertia matrix I_b with respect to its body reference frame are as follows:

$$\dot{\zeta} = v_I$$

$$m\dot{v}_I = f_I = mge_D - TR^\mathsf{T} e_z \tag{8}$$

$$I_b \dot{\omega} = -\omega \times I_b \omega + \Gamma \tag{9}$$

$$\dot{\eta} = J(\eta)\omega. \tag{10}$$

Which results in a cascade structure, where the inertial force (f_I) depends on the orientation as described by the Euler angle η. (10) relates the frame angular velocity ω to the rate change of the Euler angle $\dot{\eta}$ which depends on the frame control torque $\Gamma^\mathsf{T} = [\gamma_x, \gamma_y, \gamma_z]^\mathsf{T}$. Each control torque is applied about each corresponding frame axis and positive torque follows the right hand rule. This cascade structure is an overall nonpassive structure which has many passive subsystems. The overall approach in designing a controller for this system will be to close the loop on the passive subsystems with proportional feedback in order to design a passive attitude controller. The closed-loop dynamics of the resulting attitude control system can be neglected in order to justify an inertial position ($\zeta^\mathsf{T} = [\zeta_N, \zeta_E, \zeta_D]^\mathsf{T}$) controller. In the inertial frame, ζ_N is the distance from the origin along the e_N axis, ζ_E is the distance from the origin along the e_E

axis, and ζ_D is the distance from the origin along the e_D axis. Note that $\zeta_D < 0, \dot{\zeta}_D < 0$ corresponds to the UAV flying above and away from the inertial origin.

Using the shorthand notation $c_x = \cos x$ and $s_x = \sin x$, the rotation matrix R is related to the Euler angles as follows [15, Section 5.6.2]:

$$R = \begin{bmatrix} c_\theta c_\psi & c_\theta s_\psi & -s_\theta \\ s_\phi s_\theta c_\psi - c_\phi s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & c_\theta s_\phi \\ c_\phi s_\theta c_\psi + s_\phi s_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi & c_\theta c_\phi . \end{bmatrix} \quad (11)$$

The matrix $J(\eta)$ is the inverse of the Euler angle rates matrix $[E'_{123}(\eta)]^{-1}$ [15, Section 5.6.4] such that

$$J(\eta) = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \frac{\sin\phi}{\cos\theta} & \frac{\cos\phi}{\cos\theta} \end{bmatrix}. \quad (12)$$

Simulations indicate that when $|p|, |q|, |r| < 0.5$, and the pitch and roll are limited to the range of $[-\frac{\pi}{4}, \frac{\pi}{4}]$ then $H : \omega \to \eta$ is inside the sector $[-.004, \infty]$. Other attitude parametrizations such as the modified Rodrigues parameters are passive with angular velocity as the input [11].

The relationship between inertial acceleration, control thrusts, and the Euler angles is

$$m\dot{v}_I = \begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix} + f_{Ic}, \;\; f_{Ic} = -T \begin{bmatrix} c_\phi s_\theta c_\psi + s_\phi s_\psi \\ c_\phi s_\theta s_\psi - s_\phi c_\psi \\ c_\theta c_\phi \end{bmatrix} \quad (13)$$

in which f_{Ic} denotes the inertial control force, $T = \sum_{i=1}^{4} T_i$ is the total thrust applied by each rotor T_i, $i \in \{1, 2, 3, 4\}$. Ignoring blade flapping effects, the control torques Γ and total thrust T have the following relationship:

$$\begin{bmatrix} \gamma_x \\ \gamma_y \\ \gamma_z \\ T \end{bmatrix} = \begin{bmatrix} 0 & -\delta & 0 & \delta \\ \delta & 0 & -\delta & 0 \\ -K_t & K_t & -K_t & K_t \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} \quad (14)$$

in which δ is the distance from the center of gravity for each rotor of the UAV along the x and y body frame axis and K_t captures the relationship between rotor velocity and corresponding torques applied about the z-axis. If $\delta K_t \neq 0$ then the matrix in (14) is invertible and can be used to compute T_i from T and Γ. Since $J(\eta)$ does not depend on ψ then we can control yaw independent of ζ. With yaw as a free variable and using a small angle assumption on ϕ and θ we have

$$\frac{f_{Ic}}{-T} \approx \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} s_\psi & c_\psi \\ -c_\psi & s_\psi \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \phi \\ \theta \end{bmatrix}. \quad (15)$$

Therefore the desired inertial control command $f_{Ic}^{\mathsf{T}} = [f_{Icx}, f_{Icy}, f_{Icz}]$, will be used to compute ϕ_{set} and θ_{set} such that

$$\begin{bmatrix} \phi_{\mathsf{set}} \\ \theta_{\mathsf{set}} \end{bmatrix} = \begin{bmatrix} s_\psi & -c_\psi \\ c_\psi & s_\psi \end{bmatrix} \begin{bmatrix} \frac{f_{Icx}}{f_{Icz}} \\ \frac{f_{Icy}}{f_{Icz}} \end{bmatrix}. \quad (16)$$

The lag between each motors thrust command T_{c-i} ($i \in \{1, 2, 3, 4\}$) and the actual thrust applied by each rotor in terms of the time constant τ is

$$T_i(s) = \frac{T_{c-i}(s)}{\tau s + 1}. \quad (17)$$

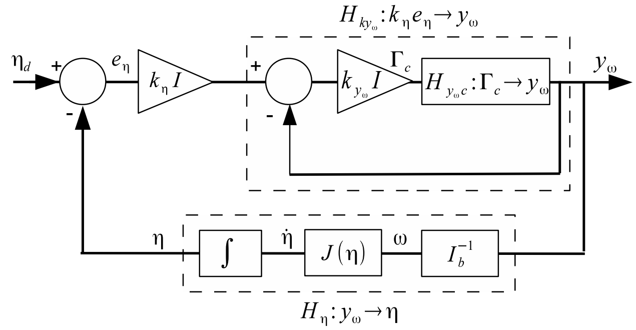

Fig. 4. Proposed attitude control system.

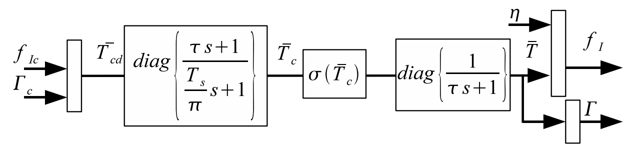

Fig. 5. Relationship between desired inertial control force (f_{Ic}) and control torque Γ_c to actual inertial force f_I and torque Γ.

IV. Control Implementation

A. Attitude Control System

There are numerous Lyapunov (and/or) passivity based approach to control attitude [3], [4], [12], [17]–[21]. We will follow the passivity based approach to control attitude by considering the scaled output $y_\omega = I_b\omega$ and assuming the inertia matrix is constant and invertable such that $\omega = I_b^{-1} y_\omega$. Under these assumptions $H : \Gamma \to y_\omega$ is passive.

Theorem 2: Any rigid body with a full-rank inertia matrix such that I_b^{-1} exists and $\dot{I}_b = 0$ whose dynamics satisfying the Euler-Lagrange equation (9) (in which ω, $\Gamma \in \mathbb{R}^3$ and $y_\omega = I_b\omega$) is a lossless passive system $H : \Gamma \to y_\omega$.

If $\Gamma = k_{y_\omega}(k_\eta e_\eta - y_\omega)$ then the closed loop system $H_{ky_\omega} : k_\eta e_\eta \to y_\omega$ is inside the sector $[0, 1]$. In addition if $H_\eta y_\omega \to \eta$ is passive then Theorem 1-i can be used to show that the the closed loop system depicted in Fig. 4 is stable. However, simulations inidcate that $H_\eta : y_\omega \to \eta$ is inside the sector $[a_{y_\omega - \eta}, \infty]$ ($a_{y_\omega - \eta} = -\lambda_{\min}^{-1}(I_b).004$).

Corollary 1 is used to address the fact that $H_\eta : y_\omega \to \eta$ is not passive (but stable). It allows us to accurately assume that the resulting cascade of $H_{ky_\omega} : k_\eta e_\eta \to y_\omega$ and $H_\eta : y_\omega \to \eta$ (denoted as $H_\eta H_{ky_\omega} : k_\eta e_\eta \to \eta$) is inside the sector $[a_\eta, \infty]$ ($-\infty < a_\eta < 0$).

Corollary 2: If the cascaded system $H_\eta H_{ky_\omega} : k_\eta e_\eta \to \eta$ is inside the sector $[a_\eta, \infty]$ ($a_\eta < 0$) and $0 < k_\eta < -\frac{1}{a_\eta}$ then the proposed closed-loop attitude control system $H_{cly_\omega} : \eta_d \to y_\omega$, $k_\omega > 0$ depicted in Fig. 4 is bounded.

The lead compensator depicted in Fig. 5 as $\mathrm{diag}\{\frac{\tau s+1}{\frac{T_s}{\pi}s+1}\}$ maximizes a_η which allows for a larger gain k_η to be realized.

B. Inertial Control System

In discussing stability for the inertial control system we will denote the system which includes the gravity compen-

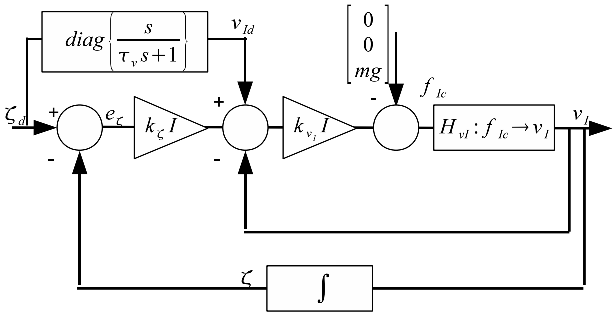

Fig. 6. Proposed inertial control system.

sation as $H_{gcomp} : k_{v_I} e_{v_I} \to v_I$ in which

$$f_{Ic} = k_{v_I} e_{v_I} - \begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix}.$$

If $f_I = (f_{Ic} + mge_D)$ then $m\dot{v}_I = k_{v_I} e_{v_I}$, therefore $H_{gcomp} : k_{v_I} e_{v_I} \to v_I$ is passive.

Corollary 3: The proposed closed-loop inertial control system $H_{cl\zeta} : \zeta_d \to v_I$, $k_{v_I} > 0$ depicted in Fig. 6 is bounded if the gravity-compensated system $H_{gcomp} : k_{v_I} e_{v_I} \to v_I$ is passive, since $\int : v_I \to \zeta$ is passive.

Finally, when $H_{gcomp} : k_{v_I} e_{v_I} \to v_I$ is cascaded with an integrator in which

$$\zeta = \int_0^T v_I dt,$$

we denote this cascaded system as $\int H_{gcomp} : k_{v_I} e_{v_I} \to \zeta$ and state the following corollary.

Corollary 4: The proposed closed-loop inertial control system $H_{cl\zeta} : \zeta_d \to v_I$, $k_{v_I} > 0$ depicted in Fig. 6 is bounded if the cascaded system $\int H_{gcomp} : k_{v_I} e_{v_I} \to \zeta$ is inside the sector $[a_\zeta, \infty]$ $(a_\zeta < 0)$ and $0 < k_\zeta < -\frac{1}{a_\zeta}$.

V. SIMULATION AND VERIFICATION

A. Verification Sector Bounds Through Simulation

The sector relations from the previously given corollaries allow verification of the sector bounds in a simulation environment. Definition 3 gives the required equation in terms of the input x and output $y = Hx$ and sector limits $[a, b]$.

$$\|y_T\|_2^2 - (a+b)\langle y, x\rangle_T + ab\|x_T\|_2^2 \le 0$$

To find bounds for a given input signal x and output signal y, consider the following:

$$\lim_{b\to\infty} \left\{ \frac{1}{b}\|y_T\|_2^2 - \frac{a+b}{b}\langle y, x\rangle_T + a\|x_T\|_2^2 \right\} \le 0$$
$$\Rightarrow -\langle y, x\rangle_T + a\|u_T\|_2^2 \le 0$$
$$\Rightarrow -\infty < a < \frac{\langle y, x\rangle_T}{\|x_T\|_2^2} \quad (18)$$

(18) is computed during the system simulation once $\|x_T\|_2^2 > \epsilon_{\min} > 0$. $\epsilon_{\min}$ is selected during runtime by the engineer in order to account for numerical effects related to dividing by a number close to zero. The sector coefficient, a, should be computed for the case when $y(0) = 0$.

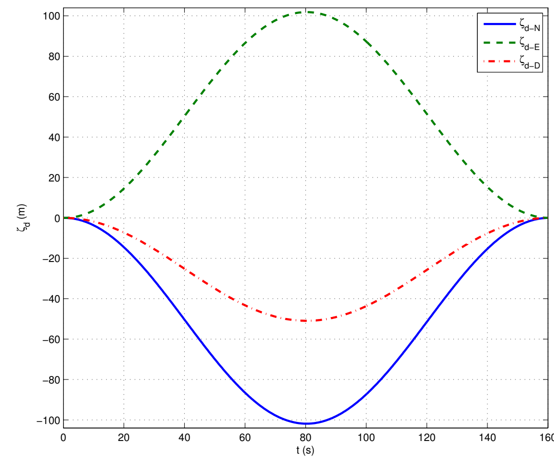

Fig. 7. Desired test flight trajectories for STARMAC and Hummingbird.

B. Simulation Parameters

Detailed models for both the STARMAC and Hummingbird quadrotor aircrafts were developed in which their physical parameters are summarized in Table I. In addition the moment of inertia matrix with respect to the body frame is $I_b = \text{diag}\{27.59, 27.71, 48.06\}$ (g·m^2) for the STARMAC and $I_b = \text{diag}\{2.32, 2.32, 4.41\}$ (g·m^2) for the Hummingbird aircraft. The sampling rates for the STARMAC $T_s = .02$ seconds and the Hummingbird $T_s = .01$ seconds. The control gains and the corresponding sector coefficients a are listed in Table II.

TABLE I
QUADROTOR PHYSICAL PARAMETERS

Aircraft	m (kg)	δ (m)	K_t (m)	τ (s)	$T_{\max}$ (N)
STARMAC	1.8263	.3048	.47	0.1	9.8
Hummingbird	0.489	.1715	.0245	.05	3.7

TABLE II
QUADROTOR CONTROL GAINS

(k_*, a_*)	STARMAC	Hummingbird
$(k_{y_\omega}, a_{y_\omega})$	(10.4,-.04)	(54,-.01)
(k_η, a_η)	(.314,-2.5)	(.0535,-18)
(k_{v_I}, a_{v_I})	(1,-.03)	(1,-.04)
(k_ζ, a_ζ)	($\frac{3}{2}$,-.62)	($\frac{5}{3}$,-.59)

The following set of figures illustrates a nominal test flight in which $\psi_r(t) = \pi \sin(.03125t)$ in which the sector stability conditions are satisfied. Fig. 7 plot the desired inertial trajectory for ζ with respect to time for the STARMAC and Hummingbird aircrafts. Fig. 8 and Fig. 9 depict the position tracking error for the respective STARMAC and Hummingbird aircrafts.

VI. CONCLUSIONS

We have shown a way to design effective control systems for quadrotor aircraft. These vehicles provide extremely challenging controller design problems; however, breaking the

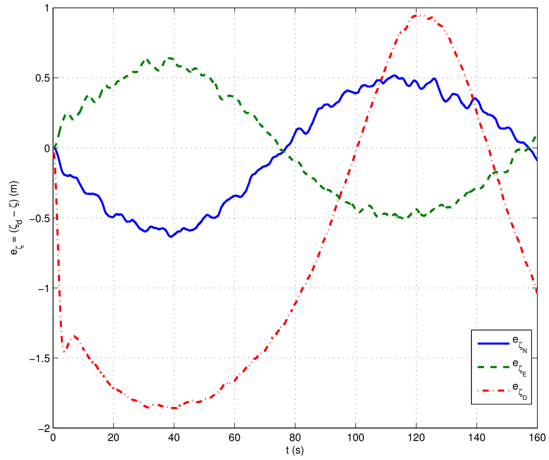

Fig. 8. Tracking error (STARMAC) for test flight.

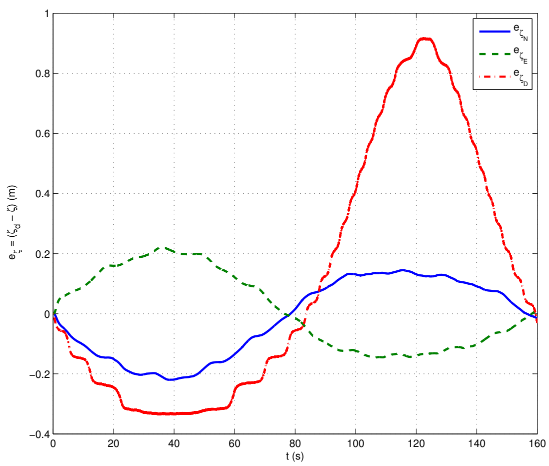

Fig. 9. Tracking error (Hummingbird) for test flight.

system down into passive components (i.e., treating inertial and attitude control separately) allows us to propose the use of simple yet effective PD controllers. We also showed and verified that yaw can be controlled independently of the desired inertial position. Furthermore, we can use a basic lead compensator to account for non ideal lag effects due to thrust. By limiting the command range for pitch and roll we can naturally address actuator saturation issues. System stability can then be verified over a fairly large range of operational conditions by means of Corollary 1. Unfortunately, for higher frequency setpoint content k_ζ will not satisfy Corollary 1 – however, the quadrotor aircraft remains stable in simulation. This simulation result emphasizes that Corollary 1 is a sufficient condition for stability. Recent results involving mixed passivity and small gain stability results [22], [23] may provide a weaker set of conditions which will satisfy our gains settings for our simulations which don't satisfy Corollary 1.

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